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DRIFT, DEFORMATION, AND DIFFUSION OF A PLASMA IN A MAGNETIC FIELD

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DRIFT, DEFORMATION, AND DIFFUSION OF A PLASMA IN A MAGNETIC FIELD

by

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32

ABSTRACT

The theoretical problem of a weakly ionized, constant temperature, three particle plasma in an externally generated magnetic field is reformulated by transforming the set of 14 macroscopic plasma equations (continuity and momentum equations for ions and electrons plus Maxwell's equations) in 14 unknowns (ion and electron number densities and velocities plus the effective electric and magnetic fields) into an equivalent set of four integral equations in four unknowns. In the course of this transformation, it is shown that plasma behavior can be interpreted in terms of three ambipolar processes: drift, deformation, and diffusion. Plasma diffusion is characterized by two diffusion coefficients: the usual Schottky formula applying in the direction parallel to the effective magnetic field and a new expression for the ambipolar transverse diffusion coefficient applying in directions perpendicular to the effective magnetic field. The new ambipolar coefficient differs markedly from the familiar ambipolar coefficient associated with the names of Bickerton, Lehnert, Holway, Allis and Buchsbaum; and, in general, it gives values for the transverse diffusion coefficient which are two orders of magnitude larger than those given by the latter. It is concluded that ambipolar diffusion can produce a transverse diffusion coefficient large enough to account for measured diffusion rates.

AUTHORIZATION

This report describes research performed at Riverside Research Institute and was prepared by R. Monroe.

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TABLE OF CONTENTS

I. INTRODUCTION

A coherent theoretical description of a weakly ionized plasma in a magnetic field has been an elusive goal of plasma physicists for many years. Theoretical studies of three particle plasmas (electrons, ions, and neutrals) go back at least to 1939[1], and these were preceded by studies of two particle plasmas (electrons and neutrals) which began in 1906[2,3,4]. Since this early work, scores of theories have been proposed to account for the complex and frequently surprising behavior of a plasma when it interacts with an externally generated magnetic field. Yet today, few of these can describe even a limited range of the observed phenomena, and none can give a unified account of the fundamental processes at work in such a plasma.

Some of the reasons for the present fragmented state of the theory can be found in the many restrictive assumptions and approximations which have been imposed on the problem in the name of mathematical expediency. These restrictions have led investigators to groups of isolated and often contradictory theoretical results which fail to provide a coherent picture of the plasma. The best illustration of this situation is the collection of expressions for the transverse (to the magnetic field) diffusion coefficient D_t of a three particle plasma. At least five expressions for D_t have been obtained under various assumptions:

$$D_t = D_e \Big/ \left(1 + \left(\frac{3\mu_e}{4}\right)^2 \overline{B} \cdot \overline{B}\right) \qquad \text{Chapman and Cowling}^1 \qquad (1)$$

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[1] For numbered references, see Sec. V.

$$D_t = \alpha k T_e / e|\overline{B}| \qquad \text{Bohm et al.}^{5} \qquad (2)$$

$$D_t = D_a \Big/ \left(1 + \mu_i \mu_e \overline{B}\cdot\overline{B}\right) \qquad (3)$$

Bickerton,[6] Lehnert,[7] Allis and Buchsbaum,[8] Holway[9]

$$D_t = D_i \Big/ \left(1 + \mu_i^2 \overline{B}\cdot\overline{B}\right) \qquad (4)$$

Simon,[10] Tonks[11]

$$D_t = \frac{k\left(T_i + T_e\right)}{e\left(\mu_i + \mu_e\right)^2}\left[\mu_e^2\mu_i \Big/ \left(1 + \mu_i^2\overline{B}\cdot\overline{B}\right) + \mu_i^2\mu_e \Big/ \left(1 + \mu_e^2\overline{B}\cdot\overline{B}\right)\right] \qquad (5)$$

Simon[12]

where μ_i and μ_e are the ion and electron mobilities, D_i and D_e the intrinsic diffusion coefficients for ions and electrons, T_i and T_e the ion and electron temperatures, and $\overline{B}$ the magnetic flux density. In addition, α is a proportionality constant, k is Boltzmann's constant, e is the magnitude of the electronic charge, and D_a is the ambipolar diffusion coefficient of a plasma in the absence of a magnetic field:[13]

$$D_a = \left(\mu_i D_e + \mu_e D_i\right) \Big/ \left(\mu_i + \mu_e\right) \qquad (6)$$

In this collection, Eq. (1) was obtained by direct solution of Boltzmann's equations for the ions and electrons. One might therefore expect it to be more accurate than the others which were all obtained by less sophisticated means. In fact, however, (1) differs only slightly from the corresponding expression for a two particle plasma which Bohm et al.[14] showed was much too small to account for their laboratory observations. Equation (2) is a semi-empirical relation based on an unpublished "turbulent drain" theory of electron diffusion which Bohm et al. used to account for their observations. This expression is unique in that it predicts a $|\overline{B}|^{-1}$ variation of D_t for large magnetic

fields. Unfortunately, subsequent experiments[7,15,16] have shown a $(\overline{B}\cdot\overline{B})^{-1} = |\overline{B}|^{-2}$ variation such as predicted by all classic (non-turbulent) theories, and, thus, the validity of (2) is doubtful. Equation (3) is referred to as the ambipolar transverse diffusion coefficient because it is based in part on the assumption that ions and electrons diffuse at the same rate across the magnetic field. Unlike all the others, this expression reduces to the correct limit $D_t \rightarrow D_a$ (Eq. (6)) as $|\overline{B}| \rightarrow 0$. As will be emphasized throughout this paper, the ambipolar assumption is generally valid; however, in obtaining (3), other, more restrictive, assumptions are imposed on the particle velocities and internal fields with the result that this expression like (1) gives values for D_t which are much too small to account for experimental observations[7,16] except when $\mu_i\mu_e\overline{B}\cdot\overline{B} \rightarrow 0$. By contrast, Eqs. (4) and (5) are in good agreement with measured values of D_t. However, these expressions were obtained by abandoning ambipolar diffusion in favor of a process in which electrons are constrained to move only along magnetic field lines while ions diffuse across lines at the rates given by (4) and (5). Charge neutrality is maintained by currents at the boundaries which must therefore be conductors, either metal in the case of laboratory plasmas (Eq. (4)) or another plasma such as the ionosphere in the case of a plasma cloud released in the upper atmosphere (Eq. (5)). Thus, the derivation of (4) and (5) depends on a restricted type of particle motion and a particular set of plasma boundary conditions. According to this picture, diffusion should revert to the ambipolar type equation (3) when the plasma is contained within insulating boundaries. However, Eq. (3) invariably gives values for D_t which are much smaller than measured values even when experiments are carried out with quartz discharge tubes[16] which one would expect to provide a working approximation to a perfect insulator. Thus, the theory leaves us in a dilemma with respect to D_t.

The preceding shows that there is a clear need to reformulate the problem of a weakly ionized plasma in a magnetic field on a more rigorous basis than has been the practice in the past. The present paper is an attempt to do this by avoiding restrictive assumptions and dealing in a self-consistent manner with the macroscopic plasma equations (continuity and momentum equations for ions and electrons plus Maxwell's equations). We limit ourselves to an explicit set of assumptions which is sufficiently general to insure self-consistency and sufficiently specific to permit the reduction of the plasma equations to a smaller and more tractable set of equations. Our choice for this set of assumptions is the following:

A. Inertial forces are negligible in the ion and electron momentum equations.

B. Displacement currents are negligible in Maxwell's equations.

C. Externally maintained electric fields within the plasma are not present.

D. Ion and electron temperatures are constant.

E. Charged particle--neutral particle collision frequencies are constant.

F. The neutral particle velocity is a known constant.

G. Ion--electron collisions are negligible.

H. Ions are singly ionized and of a single species.

These are among the least restrictive of the conditions usually imposed in investigations of this problem and should not require extensive comment here. Assumption A limits our considerations to time regimes much greater than the mean times between collisions. B excludes electromagnetic wave phenomenon. C excludes

arbitrary distributions of charge within the plasma. D and E reduce particle mobilities and diffusion coefficients to constants. F means that effects of ion and electron motion on neutral particle motion will be ignored. G and H are included as matters of convenience; both could be dropped at the expense of encumbering the mathematical details. More important from the standpoint of this paper are the commonly imposed restrictions which will not be used here. Thus, contrary to the nearly universal practice at present, no restrictions will be placed on the motion of ions and electrons beyond those required by the plasma equations. Hypothetical "congruence relations"[8,9] will not be used. The treatment will be fully three dimensional in the spatial variables; there will be no assumptions of one or two dimensional particle distributions.[17] Steady-state plasmas[7,8] will not be assumed; partial derivatives with respect to time will be retained in the continuity equations and, in general, all quantities will be functions of time as well as the spatial variables. It will not be assumed that the effective magnetic field within the plasma is equal to the known external magnetic field as is the current practice. Rather, the effective field will be considered an unknown variable equal to some as yet undetermined function of the external field and the induced field generated by currents within the plasma. To do this in a self-consistent manner will require that the complete set of Maxwell's equations be used. Truncated versions of Maxwell's equations, such as those consisting solely of Poisson's equation,[18] will not be used. Similarly, no preconditions will be imposed on the electric field other than that it originate solely within and from the plasma. Cylindrically symmetric electric fields[9] will not be assumed.

With the preceding conditions understood, we have the following macroscopic equations for a three particle weakly ionized plasma in a magnetic field:

Continuity equations for ions and electrons

$$\partial n_i/\partial t + \nabla\cdot(n_i\bar{V}_i) = 0 \tag{7}$$

$$\partial n_e/\partial t + \nabla\cdot(n_e\bar{V}_e) = 0 \tag{8}$$

Momentum equations for ions and electrons

$$n_i(\bar{V}_i - \bar{V}_n) = -D_i\nabla n_i + \mu_i n_i(\bar{E} + \bar{V}_i \times \bar{B}) \tag{9}$$

$$n_e(\bar{V}_e - \bar{V}_n) = -D_e\nabla n_e - \mu_e n_e(\bar{E} + \bar{V}_e \times \bar{B}) \tag{10}$$

Maxwell's equations

$$\nabla \times \bar{E} = -\partial\bar{B}/\partial t \tag{11}$$

$$\nabla \times \bar{B} = \mu_o e(n_i\bar{V}_i - n_e\bar{V}_e) \tag{12}$$

where n_i and n_e are the particle densities of ions and electrons respectively, $\bar{V}_i$ and $\bar{V}_e$ the particle velocities of ions and electrons, $\bar{E}$ and $\bar{B}$ the electric field and magnetic flux density, μ_o the permeability of free space, and $\bar{V}_n$ the neutral particle velocity. As previously defined μ_i and μ_e are the ion and electron mobilities, and D_i and D_e are the intrinsic diffusion coefficients. D_i and D_e can be written in terms of μ_i and μ_e.

$$D_i = kT_i\mu_i/e \quad , \quad D_e = kT_e\mu_e/e \tag{13}$$

and the latter in terms of ν_i, the ion-neutral collision frequency, and ν_e, the electron-neutral collision frequency;

$$\mu_i = e/m_i \nu_i \quad , \quad \mu_e = e/m_e \nu_e \tag{14}$$

where k, e, T_i, and T_e are as previously defined and, in addition, m_i and m_e are the ion and electron masses.

Equations (7) to (12) comprise a set of 14 equations in 14 unknowns $(n_i, n_e, \overline{V}_i, \overline{V}_e, \overline{E}, \overline{B})$. In the following section, this set will be reduced to an equivalent set of seven equations in seven unknowns. In the course of the reduction, the following points will be noted:

1. $n_i = n_e$, that is, strict charge neutrality is maintained in the plasma described by Eqs. (7) to (12).
2. A self-consistent treatment cannot assume that the effective magnetic field within the plasma is constant.
3. All plasma behavior can be interpreted in terms of three ambipolar processes: drift, deformation, and diffusion.
4. The general expression for the transverse ambipolar diffusion coefficient is:

$$D_t = \left(\mu_i D_e/(1 + \mu_e{}^2 \overline{B}\cdot\overline{B}) + \mu_e D_i/(1 + \mu_i{}^2 \overline{B}\cdot\overline{B})\right)/(\mu_i + \mu_e) \tag{15}$$

where $\overline{B}$ is not necessarily constant.

In Sec. III a zeroth order solution for $n = n_i = n_e$ is obtained using initial estimates for the effective electric and magnetic fields. With the aid of this solution and by the introduction of a vector potential the set of differential equations is then transformed into a set of four integral equations in four unknowns which can be solved by successive approximations to obtain improved estimates of n, $\overline{E}$, and $\overline{B}$.

II. A REDUCED SET OF PLASMA EQUATIONS

Equations (7) to (12) will be reduced to an equivalent set of seven equations in seven unknowns by eliminating one of the continuity equations and both of the momentum equations. We first demonstrate that only one of the continuity equations can be independent by showing that $n_i = n_e$. Thus, subtracting (8) from (7), we have:

$$\partial(n_i - n_e)/\partial t + \nabla\cdot(n_i\overline{V}_i - n_e\overline{V}_e) = 0 \tag{16}$$

But from (12),

$$\nabla\cdot(n_i\overline{V}_i - n_e\overline{V}_e) = \nabla\cdot(\nabla \times \overline{B}) = 0$$

Hence, (16) reduces to

$$\partial(n_i - n_e)/\partial t = 0$$

Integrating the latter, we obtain:

$$n_i - n_e = \rho(\overline{r}) \tag{17}$$

where ρ, the static space charge density, is an <u>arbitrary</u> function of position only. Now an arbitrary distribution of static charge could only be maintained by externally applied electric fields. Such fields are excluded by Assumption (C); hence, $\rho = 0$, and $n_i = n_e$ follows. We may then define the plasma density n as follows:

$$n \equiv n_i = n_e \tag{18}$$

and dispense with one of the continuity equations.

The preceding shows that Eqs. (7) to (12) together with Assumption (C) describe a plasma in which strict charge-

neutrality is maintained. Thus, it is not necessary to introduce charge-neutrality as a separate assumption. True space charge effects for which

$$n_i - n_e = \rho(\overline{r},t) \tag{19}$$

where ρ is not an arbitrary function, cannot be treated in a self-consistent manner on the basis of the plasma equations as we have written them here. In order to treat space charge effects consistently, it is necessary to include the displacement term $+\epsilon_o \frac{\partial \overline{E}}{\partial t}$ on the right side of Eq. (12). It is the presence of this term in (12) that enables one to obtain the auxiliary relation

$$n_i - n_e = \epsilon_o \nabla \cdot \overline{E} \not\equiv 0 \tag{20}$$

by retracing the steps from (16) to (17). In spite of this, frequent attempts are made to treat plasma-generated space charge effects by postulating Eq. (20) together with (12) without including the displacement term in the latter. Whatever may be the merits of such a procedure, consistency with Maxwell's equations is not among them since, as shown above, Eq. (12) and the continuity equations lead inexorably to $n_i - n_e = 0$ and $\nabla \cdot \overline{E} = 0$ in the absence of externally-generated electric fields. As consistency is one of the main goals of the present investigation, we will not compromise it by introducing (20) into the treatment. The following will show that, besides being undesirable from the standpoint of consistency, Eq. (20) is unnecessary for an adequate theoretical description of plasma drift and diffusion.

Following Johnson and Hulburt,[19] we can now eliminate the momentum equations by solving (9) and (10) for the ion and electron velocities

$$\overline{V}_i = \left[\overline{\chi}_i + \mu_i\overline{\chi}_i \times \overline{B} + \mu_i{}^2(\overline{\chi}_i \cdot \overline{B})\overline{B} - D_i\frac{\nabla n}{n} - \mu_i D_i \frac{\nabla n \times \overline{B}}{n} - \mu_i{}^2 D_i \frac{(\nabla n \cdot \overline{B})}{n}\,\overline{B}\right] \Big/ B_\perp \qquad (21)$$

$$\overline{V}_e = \left[\overline{\chi}_e - \mu_e\overline{\chi}_e \times \overline{B} + \mu_e{}^2(\overline{\chi}_e \cdot \overline{B})\overline{B} - D_e\frac{\nabla n}{n} + \mu_e D_e \frac{\nabla n \times B}{n} - \mu_e{}^2 D_e \frac{(\nabla n \cdot \overline{B})}{n}\,\overline{B}\right] \Big/ B_e \qquad (22)$$

where

$$\overline{\chi}_i = \overline{V}_n + \mu_i\overline{E} \quad , \quad \overline{\chi}_e = \overline{V}_n - \mu_e\overline{E} \qquad (23)$$

$$B_i = 1 + \mu_i{}^2\overline{B}\cdot\overline{B} \quad , \quad B_e = 1 + \mu_e{}^2\overline{B}\cdot\overline{B} \qquad (24)$$

and substituting these expressions into Eq. (12) and either of the continuity equations. In this way, Eqs. (7) and (12) become

$$\frac{\partial n}{\partial t} + \nabla \cdot \left[\left(n\left(\overline{\chi}_i + \mu_i\overline{\chi} \times \overline{B} + \mu_i{}^2(\overline{\chi}_i \cdot \overline{B})\overline{B}\right) - D_i\nabla n - \mu_i D_i \nabla n \times \overline{B} - \mu_i{}^2 D_i (\nabla n \cdot \overline{B})\overline{B}\right) \Big/ B_i\right] = 0 \qquad (25)$$

$$\begin{aligned}\nabla \times \overline{B} = \mu_o e \Big[& n\Big(\overline{\chi}_i/B_i - \overline{\chi}_e/B_e + (\mu_i\overline{\chi}_i/B_i + \mu_e\overline{\chi}_e/B_e) \times \overline{B} \\ & + \left((\mu_i{}^2\overline{\chi}_i/B_i - \mu_e{}^2\overline{\chi}_e/B_e)\cdot\overline{B}\right)\overline{B}\Big) - (D_i/B_i - D_e/B_e)\nabla n \\ & - \left(\mu_i D_i/B_i + \mu_e D_e/B_e\right)\nabla n \times \overline{B} - \left(\mu_i{}^2 D_i/B_i - \mu_e{}^2 D_e/B_e\right)(\nabla n \cdot \overline{B})\overline{B}\Big]\end{aligned} \qquad (26)$$

The preceding together with (11) are a set of seven equations in seven unknowns $(n, \overline{E}, \overline{B})$ equivalent to the original plasma equations.

By solving (26) for $\overline{E}$ in terms of $\nabla \times \overline{B}$ and using the resulting expression to replace $\overline{E}$ in (25), the continuity equation can be rewritten in a more convenient symmetric form. Thus, we find after a lengthy calculation (see appendix)

$$\overline{E} = -\overline{V}_n \times \overline{B} + \left(\frac{D_i - D_e}{\mu_i + \mu_e}\right) \frac{\nabla n}{n} + \left(\frac{\mu_i D_e + \mu_e D_i}{\mu_i + \mu_e}\right) \frac{\nabla n \times \overline{B}}{n}$$

$$+ \left[\frac{B_i B_e}{\mu_o e (\mu_i + \mu_e)^2 n}\right] \left[\left(\frac{\mu_i}{B_i} + \frac{\mu_e}{B_e}\right) \nabla \times \overline{B} \right. \tag{27}$$

$$\left. - \left(\frac{\mu_i^2}{B_i} - \frac{\mu_e^2}{B_e}\right)(\nabla \times \overline{B}) \times \overline{B} - \left(\frac{\mu_i \mu_e (\mu_i + \mu_e)}{B_i B_e}\right)((\nabla \times \overline{B}) \cdot \overline{B})\overline{B}\right]$$

Substituting (27) into (25) and carrying out another lengthy calculation, we obtain

$$\partial n/\partial t + \nabla \cdot \left[n\overline{V}_n - D_a \nabla n + \mu_i \mu_e (\nabla \times \overline{B}) \times \overline{B}/\mu_o e(\mu_i + \mu_e)\right] = 0 \tag{28}$$

where D_a has been defined previously (Eq. (6)). This form of the continuity equation verifies an observation of Holway[20] that the assumption of strict "congruence" where[21]

$$n_i \overline{V}_i = n_e \overline{V}_e \tag{29}$$

leads to the conclusion that a magnetic field has no effect on diffusion of charged particles. For, if (29) were valid, then (12) would reduce to $\nabla \times \overline{B} = 0$ and (28) would become:

$$\partial n/\partial t + \nabla \cdot \left[n\overline{V}_n - D_a \nabla n\right] = 0 \tag{30}$$

which is just the diffusion equation for a field-free plasma with a drift velocity equal to the neutral particle velocity $\overline{V}_n$. Since (30) is clearly inappropriate for a plasma in a magnetic field, it follows that (29) cannot be valid. At the same time (28) shows that one is never completely justified in assuming that the effective magnetic field is constant within the plasma. For if $\overline{B}$ is constant, then $\nabla \times \overline{B} = 0$ and (28) reduces to (30) which again is inappropriate.

Equation (28) can be rewritten in several interesting ways. Perhaps the most illuminating of these is obtained by noting from (26) that

$$(\nabla \times \overline{B}) \times \overline{B} = \mu_o e \left[n\overline{C} \times \overline{B} - \eta \nabla n \times \overline{B} - \theta \left((\overline{B} \cdot \nabla n) \overline{B} - (\overline{B} \cdot \overline{B}) \nabla n \right) \right] \tag{31}$$

where

$$\overline{C} = \phi \overline{V}_n + \psi \overline{E} + (\psi \overline{V}_n + \beta \overline{E}) \times \overline{B} \tag{32}$$

$$\eta = D_i/B_i - D_e/B_e \tag{33}$$

$$\theta = \mu_i D_i/B_i + \mu_e D_e/B_e \tag{34}$$

$$\phi = 1/B_i - 1/B_e \tag{35}$$

$$\psi = \mu_i/B_i + \mu_e/B_e \tag{36}$$

$$\beta = \mu_i^2/B_i - \mu_e^2/B_e \tag{37}$$

and using this expression to replace $(\nabla \times \overline{B}) \times \overline{B}$ in (28). This gives

$$\partial n/\partial t + \overline{V} \cdot \nabla n + (\nabla \cdot \overline{V}) n - \nabla \cdot \left[D_t \nabla n + \theta \mu_i \mu_e (\overline{B} \cdot \nabla n) \overline{B} / (\mu_i + \mu_e) \right] = 0 \tag{38}$$

where

$$\overline{V} = \overline{V}_n + \mu_i \mu_e \overline{C} \times \overline{B} / (\mu_i + \mu_e) + \nabla \times (\eta \overline{B}) \tag{39}$$

$$D_t = (\mu_i D_e/B_e + \mu_e D_i/B_i) / (\mu_i + \mu_e) \tag{40}$$

Decomposing the gradient operator into components perpendicular and parallel to the magnetic field,

$$\nabla = \nabla_t + \overline{B}(\overline{B} \cdot \nabla) / \overline{B} \cdot \overline{B} \tag{41}$$

we can rewrite the bracketed term in (38) as follows:

$$\left[\quad \right] = D_t \nabla_t n + D_a \overline{B} (\overline{B} \cdot \nabla n) / \overline{B} \cdot \overline{B} \tag{42}$$

Equation (35) then becomes

$$\partial n/\partial t + \overline{V}\cdot\nabla n + (\nabla\cdot\overline{V})n - \nabla\cdot\left[D_t\nabla_t n + D_a\overline{B}(\overline{B}\cdot\nabla n)/\overline{B}\cdot\overline{B}\right] = 0 \tag{43}$$

While $\overline{V}$ and D_t are functions of the effective electric and magnetic fields and thus subject to various interpretations, this form of the continuity equation strongly suggests that there are basically three processes at work in the plasma: drift, deformation, and diffusion. The second term in (43) produces an overall plasma drift velocity of $\overline{V}$, the third term accounts for increases or decreases in plasma density due to contractions $(\nabla\cdot\overline{V} < 0)$ or dilations $(\nabla\cdot\overline{V} > 0)$ of the plasma, and the fourth term describes plasma diffusion perpendicular (D_t) and parallel (D_a) to the effective magnetic field. We note that these processes must be ambipolar because Eq. (43) applies to both ions and electrons. (It does not matter whether one starts with the electron continuity equation (8) or the ion continuity equation (7) as was the choice here; both of these equations lead to (43) using the method employed in this section.) We note further that D_t (Eq. (40)) approaches the correct limit D_a as $\overline{B}$ goes to zero and that for nonzero $\overline{B}$

$$D_t \simeq D_i/B_i \tag{44}$$

provided the usual ordering $\mu_e \gg \mu_i$ holds. This approximation is identical to the expression obtained by Simon (Eq. (14)), but it is not restricted by the latter's assumptions. It shows that special restrictions need not be imposed on the ion and electron motions and that ambipolar diffusion can indeed produce a transverse diffusion coefficient large enough to account for measured diffusion rates. Finally, it should be emphasized that no restrictions have been placed on $\overline{B}$ in obtaining (40) and (44). Thus, these equations can be applied on a point-by-point basis in the general case when $\overline{B}$ is not constant.

III. TRANSFORMATION OF THE PLASMA EQUATIONS INTO A SET OF INTEGRAL EQUATIONS

Equations (39), (40), and (43) provide a basic description of the plasma in terms of ambipolar drift, deformation, and diffusion; and, as such, these equations contain the most important results of the present paper. Obtaining an actual solution to (43) for the plasma-density distribution in conjunction with Maxwell's equations ((11) and (26)) remains a formidable problem which will not be attempted here. In this section we will limit ourselves to describing one approach to the problem which is currently under investigation.

This approach is based on transforming Eqs. (11), (26), and (43) into an equivalent set of integral equations which, in principle at least, is amenable to solution by successive approximations. To do this, we note first that if the electric and magnetic fields were constant say $\overline{E}_o$ and $\overline{B}_o$, then Eq. (43) would reduce to:

$$\partial n_o/\partial t + \overline{V}_o \cdot \nabla n_o - D_{to}\nabla_{to}{}^2 n_o - D_a B_o \cdot \nabla(\overline{B}_o \cdot \nabla n_o)/\overline{B}_o \cdot \overline{B}_o = 0 \quad (45)$$

where $\overline{V}_o$ and D_{to} are given by (39) and (40) with $\overline{E}_o$ and $\overline{B}_o$ replacing $\overline{E}$ and $\overline{B}$. Equations of this form can be solved exactly.[9] The solution to the initial value problem for N particles distributed in the form of a spherical Gaussian with radius R_o at $t=o$ is:

$$n_o(\overline{r},t) = \frac{N}{\pi^{3/2} L T^2} \exp\left[-\frac{(\overline{r}_{to} - \overline{V}_{to} t)^2}{T^2} - \frac{(\overline{B}_o \cdot (\overline{r} - \overline{V}_o t))^2}{\overline{B}_o \cdot \overline{B}_o L^2}\right] \quad (46)$$

where we have decomposed the position vector $\bar{r}$ and drift velocity $\bar{V}_o$ into components perpendicular and parallel to $\bar{B}_o$:

$$\bar{r} = \bar{r}_{to} + \bar{B}_o(\bar{B}_o \cdot \bar{r})/\bar{B}_o \cdot \bar{B}_o \tag{47}$$

$$\bar{V}_o = \bar{V}_{to} + \bar{B}_o(\bar{B}_o \cdot \bar{V}_o)/\bar{B}_o \cdot \bar{B}_o \tag{48}$$

and we have defined T and L as follows:

$$T^2 = 4D_{to}t + R_o{}^2 \tag{49}$$

$$L^2 = 4D_a t + R_o{}^2 \tag{50}$$

Equation (46) describes a cylindrically symmetric particle distribution moving with a constant drift velocity $\bar{V}_o$ and simultaneously diffusing perpendicular and parallel to $\bar{B}_o$ with characteristic diffusion coefficients D_{to} and D_a, respectively. The contours of constant density are ellipsoids of revolution with major axes parallel to $\bar{B}_o$. Now, in general, $\bar{E}$ and $\bar{B}$ are not constant, and thus we cannot expect (46) alone to give an adequate description of the manner in which an initially spherical plasma distribution evolves in a magnetic field. Indeed, observations of barium plasma clouds in the upper atmosphere[22] show that the ultimate disposition of an initially spherical plasma distribution is much more complicated than (46), involving the development of magnetic field-aligned striations among other things. Nevertheless, (46) can provide an acceptable description of the early time behavior of such a plasma, if $\bar{B}_o$ and $\bar{E}_o$ are properly chosen. For example, if one sets $\bar{B}_o$ equal to the ambient geomagnetic field and takes $\bar{E}_o$ to be the first term in (27)

$$\bar{E}_o = - \bar{V}_n \times \bar{B}_o \tag{51}$$

then, substituting (51) into (39), we find that the drift velocity reduces to:

$$\overline{V}_o = \overline{V}_r \tag{52}$$

and, in this case, (46) does provide a reasonable approximation to what usually happens during the first two or three mintues following a barium plasma release at heights between 100 and 300 km. That is, the plasma cloud initially tends to drift with the velocity of the ambient neutral particles while diffusing parallel and perpendicular to the geomagnetic field at rates controlled by D_a and D_{to}, respectively, the latter being determined from (40) with $\overline{B}$ set equal to the geomagnetic field at the height of release. We may interpret (51) and (52) in reverse order by saying that initially the neutral particles in colliding with the charged particles tend to drag the plasma at the neutral particle velocity $\overline{V}_n$, thereby inducing an electric field given by (51). Eventually, other fields tending to oppose the plasma motion will be induced in the cloud resulting in a smaller drift velocity and a much more complicated plasma distribution. In spite of this, the success of (46) in describing the early time behavior of barium plasma clouds suggests that this expression is a valid zeroth order solution which could be used as the first step in a successive approximation solution to Eqs. (11), (26) and (43). To facilitate such a solution, we can add $\overline{V}_o\cdot\nabla n - D_{to}\nabla_{to}{}^2 n - D_a\overline{B}_o\cdot\nabla(\overline{B}_o\cdot\nabla n)/\overline{B}_o\cdot\overline{B}_o$ to both sides of Eq. (43) and rewrite (43) as follows:

$$\partial n/\partial t + \overline{V}_o\cdot\nabla n - D_{to}\nabla_{to}{}^2 n - D_a\overline{B}_o\cdot\nabla(\overline{B}_o\cdot\nabla n)/\overline{B}_o\cdot\overline{B}_o = Q[n] \tag{53}$$

where the operator $Q[\cdot]$ is defined by

$$Q[\cdot] = \overline{V}^*\cdot\nabla[\cdot] + (\nabla\cdot\overline{V}^*)[\cdot] + \nabla\cdot\overline{F}[\cdot] \tag{54}$$

and

$$\overline{V}^* = \overline{V}_o - \overline{V} \tag{55}$$

$$\overline{F}[\cdot] = D_t\nabla_t[\cdot] - D_{to}\nabla_{to}[\cdot] + D_a\left(\frac{\overline{B}(\overline{B}\cdot\acute{\nabla}[\cdot])}{\overline{B}\cdot\overline{B}} - \frac{\overline{B}_o(\overline{B}_o\cdot\nabla[\cdot])}{\overline{B}_o\cdot\overline{B}_o}\right) \tag{56}$$

With the aid of a Green's theorem,[23] (53) can be transformed into an integral equation for n:

$$n(\overline{r},t) = n_o(\overline{r},t) + \frac{1}{4\pi}\int_0^t dt' \iiint_{-\infty}^{\infty} d\overline{r}'^3 Q[n(\overline{r}',t')]G(\overline{r}-\overline{r}',t-t') \tag{57}$$

where $n_o(\overline{r},t)$ is given by (46), $G(\overline{r},t)$ is defined by:

$$G(\overline{r},t) = \frac{H(t)}{\pi^{3/2}(4D_{to}t)(4D_a t)^{1/2}} \exp\left[-\frac{(\overline{r}_{to}-\overline{V}_{to}t)^2}{4D_{to}t} - \frac{(\overline{B}_o\cdot(\overline{r}-\overline{V}_o t))^2}{(\overline{B}_o\cdot\overline{B}_o)4D_a t}\right] \tag{58}$$

and, in the latter, $H(t)$ is the step function. The form of (57) naturally suggests a solution by successive approximations with $n_o(\overline{r},t)$ as the first approximation to n. In such a process, it is necessary to calculate the effective electric and magnetic fields at each iteration, since $Q[\cdot]$ is a function of both $\overline{E}$ and $\overline{B}$. This is most easily done when Maxwell's equations are written in integral form. Introducing the vector potential $\overline{A}$ in the usual way

$$\overline{B} = \nabla \times \overline{A} + \overline{B}_o \tag{59}$$

and employing the Coulomb gauge $\nabla\cdot\overline{A} = 0$ (since $n_i = n_e$), we find that Maxwell's equations reduce to:

$$\bar{E} = - \partial\bar{A}/\partial t + \bar{E}_o \tag{60}$$

and

$$\nabla^2\bar{A} = - \mu_o\bar{J}[n,\bar{A}] \tag{61}$$

where $\mu_o\bar{J}[n,\bar{A}]$ is given by the right side of (26) with (59) and (60) replacing $\bar{B}$ and $\bar{E}$. Equation (61) may be considered a vector Poisson equation with a formal solution[24]

$$\bar{A}(\bar{r},t) = \frac{\mu_o}{4\pi} \iiint_{-\infty}^{\infty} d\bar{r}'^3 \frac{\bar{J}[n(\bar{r}',t),\bar{A}(\bar{r}',t)]}{|\bar{r}-\bar{r}'|} \tag{62}$$

Since $\bar{J}$ depends on $\bar{A}$, as well as n, Eq. (62) is actually a vector integral equation (equivalent to three scalar integral equations) which together with (57) comprises a coupled set of nonlinear integral equations in the unknowns n and $\bar{A}$. This set is exactly equivalent to the original set of macroscopic plasma equations, and its solutions can be used with Eqs. (18), (21), (22), (59) and (60) to recover complete solutions $(n_i,n_e,\bar{E},\bar{B},\bar{V}_i,\bar{V}_e)$ to Eqs. (7) to (12). We have thus succeeded in replacing our original set of 14 nonlinear differential equations in 14 unknowns with an equivalent set of four nonlinear integral equations in four unknowns.

The form of Eqs. (57) and (62) suggests a solution by successive approximations with $n = n_o$ and $\bar{A} = 0$ as initial estimates. The success of this approach will depend on the proper choice of $\bar{B}_o$ and $\bar{E}_o$ among other things. In the case of barium plasmas in the upper atmosphere, preliminary calculations using the ambient geomagnetic field for $\bar{B}_o$ and $-\bar{V}_n \times \bar{B}_o$ for $\bar{E}_o$ are encouraging.

IV. DISCUSSION

In the preceding sections we have reformulated the theoretical problem of a three particle plasma in a magnetic field by reducing the set of macroscopic plasma equations to a smaller equivalent set of equations. One of these, the continuity equation (43), consists of terms describing three ambipolar processes: drift, deformation, and diffusion. From this equation, two characteristic diffusion coefficients emerge: the Schottky formula (6) applying for diffusion parallel to the magnetic field and Eq. (40) applying for diffusion perpendicular to the magnetic field. Equation (40) differs markedly from the currently accepted expression for the transverse ambipolar diffusion coefficient (3), and, in general, (40) will give a much larger value for D_t. To show this, we form the ratio R of (40) to (3) as follows:

$$R = \left(\frac{\mu_i D_e}{B_e} + \frac{\mu_e D_i}{B_i}\right)\left(\frac{1}{\mu_i + \mu_e}\right) \Bigg/ \left(\frac{D_a}{1 + \mu_i \mu_e \overline{B}\cdot\overline{B}}\right).$$

$$\simeq (1 + \mu_i \mu_e \overline{B}\cdot\overline{B})/2(1 + \mu_i^2 \overline{B}\cdot\overline{B})$$

In most cases of interest, $\mu_i \mu_e \overline{B}\cdot B \gg 1$ and $\mu_i^2 B\cdot B \gg 1$; hence,

$$R \simeq \mu_e / 2\,\mu_i .$$

Or, using (14),

$$R \simeq m_i \nu_i / 2 m_e \nu_e .$$

For an O_2^+ plasma,[9] $m_i/m_e = 6 \times 10^4$ and $\nu_i/\nu_e = 0.5\ 10^{-2}$. Hence, $R = 150$. That is, for a typical plasma, the transverse

diffusion coefficient computed from (40) will be larger by two orders of magnitude than that computed from (3). Since the larger diffusion coefficient is in substantial agreement with measured values of D_t and since this coefficient is independent of any boundary conditions, it follows that Eq. (40) is a means of escaping the dilemma described in the introduction.

Equation (40) characterizes transverse diffusion in a plasma wherever charge neutrality is maintained as prescribed by Eqs. (7) to (12). Indeed, one may interpret (40) as characterizing the diffusive properties of a magnetized plasma consisting of charged particles which are constrained to move in such a way that the average net charge per unit volume is always zero. One would not necessarily expect this expression to apply in regions where significant deviations from charge neutrality occur, such as in the positive ion sheath which forms when a conductor is immersed in the plasma. As was pointed out in Sec. II, this more general case can be treated in a self-consistent manner only by including the displacement term in Maxwell's equations. Nevertheless, in most cases of interest, charge-neutrality is maintained throughout the greater part of the volume occupied by the plasma, and, in these cases, (40) will provide a means of computing D_t.

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VI. APPENDIX

DERIVATION OF EQUATION (27)

With the aid of (23), Eq. (26) can be rewritten as follows:

$$\nabla \times \overline{B} = \mu_o e[n(\phi \overline{V}_n + \psi \overline{E} + (\psi \overline{V}_n + \beta \overline{E}) \times \overline{B} + ((\beta \overline{V}_n + \zeta \overline{E}) \cdot \overline{B})\overline{B}$$

$$- \eta \nabla n - \theta \nabla n \times \overline{B} - \sigma(\nabla n \cdot \overline{B})\overline{B}] \tag{A1}$$

where:

$$\phi = 1/B_i - 1/B_e \tag{A2}$$

$$\psi = \mu_i/B_i + \mu_e/B_e \tag{A3}$$

$$\beta = \mu_i{}^2/B_i - \mu_e{}^2/B_e \tag{A4}$$

$$\zeta = \mu_i{}^3/B_i + \mu_e{}^3/B_e \tag{A5}$$

$$\eta = D_i/B_i - D_e/B_e \tag{A6}$$

$$\theta = \mu_i D_i/B_i + \mu_e D_e/B_e \tag{A7}$$

$$\sigma = \mu_i{}^2 D_i/B_i - \mu_e{}^2 D_e/B_e \tag{A8}$$

Rearranging (A1), we have:

$$\psi \overline{E} + \beta \overline{E} \times \overline{B} + \zeta(\overline{E} \cdot \overline{B})\overline{B} = \overline{\Omega} \tag{A9}$$

where:

$$\bar{\Omega} = \frac{\nabla \times \bar{B}}{\mu_o en} + \eta \frac{\nabla n}{n} + \theta \frac{\nabla n \times \bar{B}}{n} + \sigma\left(\frac{\nabla n \cdot \bar{B}}{n}\right)\bar{B}$$

$$- \phi\bar{V}_n - \psi\bar{V}_n \times \bar{B} - \beta(\bar{V}_n \cdot \bar{B})\bar{B} \tag{A10}$$

Hence,

$$\psi\bar{E} = \bar{\Omega} - \beta\bar{E} \times \bar{B} - \zeta(\bar{E}\cdot\bar{B})\bar{B} \tag{A11}$$

Forming the dot product of (A11) with $\bar{B}$, we obtain:

$$\psi\bar{E}\cdot\bar{B} = \bar{\Omega}\cdot\bar{B} - \zeta(\bar{E}\cdot\bar{B})(\bar{B}\cdot\bar{B}) \tag{A12}$$

Hence, solving for $\bar{E}\cdot\bar{B}$

$$\bar{E}\cdot\bar{B} = (\bar{\Omega}\cdot\bar{B})/(\psi + \zeta\bar{B}\cdot\bar{B}) \tag{A13}$$

Similarly, taking the cross product of (A11) with $\bar{B}$,

$$\psi\bar{E} \times \bar{B} = \bar{\Omega} \times \bar{B} - \beta(\bar{E}\cdot\bar{B}) \times \bar{B} \tag{A14}$$

or

$$\psi\bar{E} \times \bar{B} = \bar{\Omega} \times \bar{B} - \beta(\bar{B}(\bar{E}\cdot\bar{B}) - \bar{E}(\bar{B}\cdot\bar{B})) \tag{A15}$$

Substituting (A13) into (A15), we obtain:

$$\bar{E} \times \bar{B} = \left[\bar{\Omega} \times \bar{B} - \beta\left(\bar{B}\left(\frac{\bar{\Omega}\cdot\bar{B}}{\psi + \zeta\bar{B}\cdot\bar{B}}\right) - \bar{E}(\bar{B}\cdot\bar{B})\right)\right]\Big/\psi \tag{A16}$$

And, substituting (A13) and (A16) into (A11), we obtain:

$$\psi\bar{E} = \bar{\Omega} - \beta\left[\bar{\Omega} \times \bar{B} - \beta\left(\frac{(\bar{\Omega}\cdot\bar{B})\bar{B}}{\psi + \zeta\bar{B}\cdot\bar{B}} - \bar{E}(\bar{B}\cdot\bar{B})\right)\right] - \frac{\zeta(\bar{\Omega}\cdot\bar{B})\bar{B}}{\psi + \zeta\bar{B}\cdot\bar{B}}$$

Collecting terms, the preceding becomes:

$$(\psi^2 + \beta^2\overline{B}\cdot\overline{B})\overline{E} = \psi\overline{\Omega} - \beta\overline{\Omega} \times \overline{B} + \left(\frac{\beta^2 - \zeta\psi}{\psi + \zeta\overline{B}\cdot\overline{B}}\right)(\overline{\Omega}\cdot\overline{B})\overline{B} \tag{A17}$$

Now

$$\psi^2 + \beta^2\overline{B}\cdot\overline{B} = \left(\frac{\mu_i}{B_i}\right)^2 + \frac{2\mu_i\mu_e}{B_iB_e} + \left(\frac{\mu_e}{B_e}\right)^2 + \frac{\mu_i{}^2(B_i - 1)}{B_i{}^2} - \frac{2\mu_i{}^2\mu_e{}^2\overline{B}\cdot\overline{B}}{B_iB_e}$$
$$+ \frac{\mu_e{}^2(B_e - 1)}{B_e{}^2} = \frac{(\mu_i + \mu_e)^2}{B_iB_e} \tag{A18}$$

And

$$\frac{\beta^2 - \zeta\psi}{\psi + \zeta\overline{B}\cdot\overline{B}} = \frac{\left(\frac{\mu_i{}^2}{B_i} - \frac{\mu_e{}^2}{B_e}\right)^2 - \left(\frac{\mu_i{}^3}{B_i} + \frac{\mu_e{}^3}{B_e}\right)\left(\frac{\mu_i}{B_i} + \frac{\mu_e}{B_e}\right)}{\frac{\mu_i}{B_i} + \frac{\mu_e}{B_e} + \left(\frac{\mu_i{}^3}{B_i} + \frac{\mu_e{}^3}{B_e}\right)\overline{B}\cdot\overline{B}}$$

$$= - \frac{\mu_i\mu_e(2\mu_i\mu_e + \mu_e{}^2 + \mu_i{}^2)}{B_iB_e(\mu_i + \mu_e)} = \frac{\mu_i\mu_e(\mu_i + \mu_e)}{B_iB_e} \tag{A19}$$

With (A18) and (A19), (A17) becomes:

$$E = \frac{B_iB_e\psi\overline{\Omega}}{(\mu_i + \mu_e)^2} - \frac{B_iB_e\beta\overline{\Omega}\times\overline{B}}{(\mu_i + \mu_e)^2} - \frac{\mu_i\mu_e(\overline{\Omega}\cdot\overline{B})\overline{B}}{(\mu_i + \mu_e)} \tag{A20}$$

Or

$$\overline{E} = \frac{B_iB_e\psi}{(\mu_i + \mu_e)^2}\left(\frac{\nabla\times\overline{B}}{\mu_o en} + \frac{\nabla n}{n} + \theta\,\frac{\nabla n\times\overline{B}}{n} + \sigma\,\frac{(\nabla n\cdot\overline{B})\overline{B}}{n} - \phi\overline{V}_n - \psi\overline{V}_n\right.$$
$$\left.\times\,\overline{B} - \beta(\overline{V}_n\cdot\overline{B})\overline{B}\right) - \frac{B_iB_e\beta}{(\mu_i + \mu_e)^2}\left(\frac{(\nabla\times\overline{B})\times\overline{B}}{\mu_o en} + \eta\,\frac{\nabla n\times\overline{B}}{n}\right.$$
$$\left. + \theta\,\frac{((\nabla n\cdot\overline{B})\overline{B} - \nabla n(\overline{B}\cdot\overline{B}))}{n} - \phi\overline{V}_n \times \overline{B} - \psi(\overline{B}\cdot\overline{V}_n)\overline{B} + \psi(\overline{B}\cdot\overline{B})\overline{V}_n\right)$$
$$- \frac{\mu_i\mu_e}{(\mu_i + \mu_e)}\left(\frac{(\nabla\times\overline{B})\cdot\overline{B}}{\mu_o en} + \eta\,\frac{(\nabla n\cdot\overline{B})}{n} + \sigma\,\frac{(\overline{B}\cdot\overline{B})(\nabla n\cdot\overline{B})}{n} - \phi(\overline{V}_n\cdot\overline{B})\right.$$
$$\left. - \beta(\overline{B}\cdot\overline{B})(\overline{V}_n\cdot\overline{B})\right)\overline{B}$$

Collecting terms

$$\overline{E} = \frac{B_iB_e\psi\nabla\times\overline{B}}{(\mu_i+\mu_e)^2\mu_o en} - \frac{B_iB_e\beta(\nabla\times\overline{B})\times\overline{B}}{(\mu_i+\mu_e)^2\mu_o en} - \frac{\mu_i\mu_e((\nabla\times\overline{B})\cdot\overline{B})\overline{B}}{(\mu_i+\mu_e)\mu_o en}$$
$$+ \frac{B_iB_e(\psi\eta+\beta\theta\overline{B}\cdot\overline{B})\nabla n}{(\mu_i+\mu_e)^2 n} + \frac{B_iB_e(\psi\theta-\beta\eta)\nabla n\times\overline{B}}{(\mu_i+\mu_e)^2 n}$$
$$+ \left(\frac{B_iB_e(\psi\sigma-\beta\theta)}{(\mu_i+\mu_e)^2} - \frac{\mu_i\mu_e(\eta+\sigma(\overline{B}\cdot\overline{B}))}{(\mu_i+\mu_e)}\right)\frac{(\nabla n\cdot\overline{B})\overline{B}}{n} - \frac{B_iB_e\psi(\phi+\beta\overline{B}\cdot\overline{B})\overline{\nabla}n}{(\mu_i+\mu_e)^2}$$
$$- \frac{B_iB_e(\psi[\text{illegible}]-\beta\phi)\overline{\nabla}n\times\overline{B}}{(\mu_i+\mu_e)^2} + \frac{\mu_i\mu_e}{(\mu_i+\mu_e)}\left(\phi+\beta(\overline{B}\cdot\overline{B})\right)(\overline{\nabla}_n\cdot\overline{B})\overline{B} \qquad \text{(A21)}$$

Now

$$\frac{B_iB_e(\psi\eta+\beta\theta\overline{B}\cdot\overline{B})}{(\mu_i+\mu_e)^2} = \frac{B_iB_e}{(\mu_i+\mu_e)^2}\left[\frac{\mu_iD_i}{B_i^2} - \frac{\mu_eD_e}{B_e^2} + \frac{\mu_eD_i}{B_eB_i} - \frac{\mu_iD_e}{B_eB_i}\right.$$
$$\left. + \frac{\mu_iD_i}{B_i^2}(B_i-1) - \frac{\mu_eD_e}{B_e^2}(B_e-1) - \frac{\mu_iD_i}{B_iB_e}(B_e-1) + \frac{\mu_eD_e}{B_iB_e}(B_i-1)\right]$$
$$= \frac{\mu_eD_i-\mu_iD_e+\mu_iD_i-\mu_eD_e}{(\mu_i+\mu_e)^2} = \frac{D_i-D_e}{\mu_i+\mu_e} \qquad \text{(A22)}$$

And

$$\frac{B_iB_e(\psi\theta-\beta\eta)}{(\mu_i+\mu_e)^2} = \frac{B_iB_e}{(\mu_i+\mu_e)^2}\left[\frac{\mu_e\mu_iD_i+\mu_i\mu_eD_e+\mu_i^2D_e+\mu_e^2D_i}{B_iB_e}\right]$$
$$= \frac{\mu_iD_e+\mu_eD_i}{\mu_i+\mu_e} \qquad \text{(A23)}$$

Also

$$\frac{B_iB_e(\psi\sigma-\beta\theta)}{(\mu_i+\mu_e)^2}-\frac{\mu_i\mu_e(\eta+\sigma(\overline{B}\cdot\overline{B}))}{\mu_i+\mu_e}$$

$$=\frac{1}{(\mu_i+\mu_e)^2}\Big[\mu_e\mu_i{}^2D_i-\mu_i\mu_e{}^2D_e+\mu_e{}^2\mu_iD_i-\mu_i{}^2\mu_eD_e$$
$$-\mu_i\mu_e(\mu_i+\mu_e)(\eta+\sigma\overline{B}\cdot\overline{B})\Big]$$

$$=\frac{\mu_i\mu_e}{\mu_i+\mu_e}\left[D_i-D_e-\frac{D_i}{B_i}+\frac{D_e}{B_e}-\left(\frac{\mu_i{}^2D_i}{B_i}-\frac{\mu_e{}^2D_e}{B_e}\right)\overline{B}\cdot\overline{B}\right]$$

$$=\frac{\mu_i\mu_e}{\mu_i+\mu_e}\left[D_i-D_e-\frac{D_i}{B_i}+\frac{D_e}{B_e}-\frac{(B_i-1)D_i}{B_i}+\frac{(B_e-1)D_e}{B_e}\right]=0 \qquad \text{(A24)}$$

And

$$\frac{B_iB_e\psi(\phi+\beta\overline{B}\cdot\overline{B})}{(\mu_i+\mu_e)^2}=\frac{B_iB_e\psi}{(\mu_i+\mu_e)^2}\left(\frac{1}{B_i}-\frac{1}{B_e}+\left(\frac{\mu_i{}^2}{B_i}-\frac{\mu_e{}^2}{B_e}\right)\overline{B}\cdot\overline{B}\right)$$

$$=\frac{B_iB_e\psi}{(\mu_i+\mu_e)^2}\left(\frac{1}{B_i}-\frac{1}{B_e}+\frac{B_i-1}{B_i}-\frac{B_e-1}{B_e}\right)=0 \qquad \text{(A25)}$$

Furthermore

$$\frac{B_iB_e(\psi^2+\beta\phi)}{(\mu_i+\mu_e)^2}=\frac{B_iB_e}{(\mu_i+\mu_e)^2}=\left(\frac{\mu_i{}^2}{B_i{}^2}-\frac{\mu_e{}^2}{B_e{}^2}-\frac{2\mu_i\mu_e}{B_iB_e}\right.$$
$$\left.-\frac{\mu_i{}^2}{B_i{}^2}-\frac{\mu_e{}^2}{B_e{}^2}+\frac{\mu_i{}^2}{B_iB_e}+\frac{\mu_e{}^2}{B_iB_e}\right)$$

$$=\frac{B_iB_e}{(\mu_i+\mu_e)^2}\left(\frac{\mu_i{}^2+\mu_e{}^2+2\mu_i\mu_e}{B_iB_e}\right)=1 \qquad \text{(A26)}$$

With the aid of (A22), (A23), (A24), (A25) and (A26), (A21) becomes:

$$\overline{E} = \overline{V}_n \times \overline{B} + \left(\frac{D_i - D_e}{\mu_i + \mu_e}\right)\left(\frac{\nabla n}{n}\right) + \left(\frac{\mu_i D_e + \mu_e D_i}{\mu_i + \mu_e}\right)\left(\frac{\nabla n \times B}{n}\right)$$

$$+ \frac{B_i B_e \psi \nabla \times \overline{B}}{(\mu_i + \mu_e)^2 \mu_o en} - \frac{B_i B_e \beta (\nabla \times \overline{B}) \times \overline{B}}{(\mu_i + \mu_e)^2 \mu_o en} - \frac{\mu_i \mu_e ((\nabla \times \overline{B}) \cdot \overline{B})\overline{B}}{(\mu_i + \mu_e)\mu_o en}$$

Or

$$\overline{E} = -\overline{V}_n \times \overline{B} + \left(\frac{D_i - D_e}{\mu_i + \mu_e}\right)\left(\frac{\nabla n}{n}\right) + \left(\frac{\mu_i D_e + \mu_e D_i}{\mu_i + \mu_e}\right)\left(\frac{\nabla n \times \overline{B}}{n}\right)$$

$$+ \frac{B_i B_e}{(\mu_i + \mu_e)^2 \mu_o en}\left[\left(\frac{\mu_i}{B_i} + \frac{\mu_e}{B_e}\right)\nabla \times \overline{B} - \left(\frac{\mu_i^2}{B_i} - \frac{\mu_e^2}{B_e}\right)(\nabla \times \overline{B}) \times \overline{B}\right.$$

$$\left. - \frac{\mu_i \mu_e (\mu_i + \mu_e)((\nabla \times \overline{B}) \cdot \overline{B})\overline{B}}{B_i B_e}\right]$$

which is Eq. (27).